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Generalized Convex Multiplicative Programming via Quasiconcave Minimization

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Abstract. We present a new method for minimizing the sum of a convex function and a product of k nonnegative convex functions over a convex set. This problem is reduced to a k-dimensional quasiconcave minimization problem which is solved by a conical branch-and-bound algorithm. Comparative computational results are provided on test problems from the literature.

Key words: Generalized convex multiplicative programming, conical partition, global optimization, branch-and-bound.

1. Introduction

We consider the optimization of Generalized Convex Multiplicative Programming problems of the form:

$$(GCMP) \quad \min_{x \in G} \left(f(x) = f_0(x) + \prod_{i=1}^{k} f_i(x) \right)$$

where $k \geq 2$, the functions f_i are convex on $\mathbb{R}^n \, (i = 0, 1, \ldots, k)$, and G is a nonempty compact convex subset of $\mathbb{R}^n$ such that:

$$\forall x \in G \qquad f_i(x) \geq 0 \quad i = 1, 2, \ldots, k. \tag{1}$$

This problem has many applications, e.g., in microeconomics [5], VLSI chip design [19], bond portfolio optimization [8] and multiobjective programming [4].

It is well known (see e.g., Konno and Kuno [10]) that the product of convex functions need not be (quasi)convex thus the objective function f of problem $(GCMP)$ may have many local minima. Moreover, except if the function f_0 is constant and the functions $f_i (i = 1, 2, \ldots, k)$ are linear, the objective function is not quasiconcave in general.

Over the last five years, several particular cases of problem $(GCMP)$ have been investigated, especially when $k = 2$.

The Linear Multiplicative Programming problem (i.e., f_0 constant and f_1, f_2 linear) has been studied by Konno and Kuno [10] who proposed a parametric simplex method to solve it. A slightly more general method, taking advantage of a low degree of nonlinearity, has been suggested by Tuy and Tam [29]. The parametric simplex approach has then been generalized by Konno *et al.* [14] to solve a special case of the General Linear Multiplicative Programming problem (i.e., f_0 quadratic and f_1, f_2 linear). Recently, Schaible and Sodini [24] used this same approach (but with different optimality conditions) for the case f_0, f_1, f_2 linear. Two methods have been proposed for the case where f_0 is convex and f_1 and f_2 are linear: *(i)* a discrete approximation algorithm by Konno and Kuno [9]; *(ii)* a branch-and-bound algorithm by Muu and Tam [21] where the branching takes place in an interval of $\mathbb{R}$ and the bounding corresponds to a relaxation. Moreover, Kuno and Konno [15] designed a branch-and-bound method using an underestimating function for problem $(GCMP)$ where $k = 2$ (which could be easily extended for $k = 3$).

More recently, Konno *et al.* [12] considered the problem of minimizing the sum of p products of two convex functions, which includes the problem $(GCMP)$ when $k = 2$ as a particular case. They proposed a reduction to a concave minimization problem with $2p$ variables, which they solved by an outer approximation method.

Problems with up to 350 linear constraints and 300 variables have been solved when f_0, f_1, f_2 are linear by Konno *et al.* [14] and up to 130 linear constraints and 100 variables when f_0 is a convex quadratic function and f_1, f_2 are linear by Konno and Kuno [15].

For k greater than 2, only the Convex Multiplicative Programming problem $(f_0 = 0)$ has been considered. Thoai [26] proposes a reduction to the minimization of a quasiconcave function depending only on k variables. This last problem is then solved by an outer approximation method. Kuno *et al.* [17] propose another reduction to a concave minimization problem with k variables, the objective function value of which is determined by solving a convex minimization problem. Their computational experiments show that the resulting algorithm is reasonably efficient for $k \leq 4$.

Related problems such as minimization of fractional functions or minimization with a multiplicative constraint can be found in [9, 13, 14, 16, 18, 22, 23, 25, 27, 30]. For a recent review on multiplicative programming, see Konno and Kuno [11].

The purpose of this paper is to design a new algorithm to solve problem $(GCMP)$. It is organized as follows. In Section 2, we convert $(GCMP)$ to a quasiconcave minimization problem in a k-dimensional space. New bounds and branchings are discussed in Section 3. In Section 4, the results of the previous section are embedded in a conical branch-and-bound algorithm for solving low dimensional quasiconcave minimization problems. Computational results are reported in Section 5.

2. Reduction to Quasiconcave Minimization

In this section, we show how to reduce the generalized convex multiplicative problem $(GCMP)$ to a quasiconcave minimization one when functions $f_i(i = 1, 2, \ldots, k)$ are positive. It is a generalization of the reduction to a concave minimization problem, proposed by Kuno *et al.* [17], of the minimization of a product of k convex functions, i.e., problem $(GCMP)$ when f_0 is a constant function.

Whenever discussing the particular case where f_0 is a constant function, we will assume, without loss of generality, that f_0 is the null function, i.e., $f_0(x) = 0$ for all $x \in G$.

We will also assume that:

$$\forall x \in G \quad f_i(x) > 0 \quad i = 1, 2, \ldots, k. \tag{2}$$

This condition is not restrictive if we are interested in ε-optimal solutions (i.e., points with values differing from the optimal one by less than $\varepsilon > 0$) of problem $(GCMP)$. Indeed, let $\tilde{f}$ be the global minimum of $f(x)$ over G with the additional constraint $\prod_{i=1}^{k} f_i(x) = 0$. We can then show that either $\tilde{f}$ is an ε-optimal value of problem $(GCMP)$, or all optimal solutions of problem $(GCMP)$ belong to the convex set $G' = G \cap \{x : f_0(x) \leq \tilde{f} - \varepsilon\}$. Moreover, we have $\prod_{i=1}^{k} f_i(x) > 0$ for all $x \in G'$.

Note first that the value $\tilde{f}$ can be obtained through the solution of k convex problems:

$$(P_i) \quad \tilde{f}_i = \min\{f_0(x) : f_i(x) \leq 0 \text{ and } x \in G\}$$

for $i = 1, 2, \ldots, k$, i.e., $\tilde{f} = \min_{i=1,2,\ldots,k} \tilde{f}_i$.

Let x^* be an optimal solution of problem $(GCMP)$ with value f^*. If $\tilde{f}$ is not an ε-optimal value of problem $(GCMP)$ we have $\tilde{f} - f^* > \varepsilon$. Since $f_0(x^*) \leq f^*$, it follows that $f_0(x^*) < \tilde{f} - \varepsilon$, hence $x^* \in G'$.

Now let x' be a point of G' and assume that there exists an index j such that $f_j(x') = 0$. Then x' is a feasible solution of problem (P_j), hence $f_0(x') \geq \tilde{f}_j \geq \tilde{f}$. Thus x' does not satisfy the constraint $f_0(x) \leq \tilde{f} - \varepsilon$ of G', which is a contradiction. Obviously, G' is still a compact convex set.

Consequently, an ε-optimal solution of problem $(GCMP)$ with non negative functions can be easily obtained by comparing $\tilde{f}$ with the optimal (or an ε-optimal) solution of problem $(GCMP)$ with positive functions.

Let $\mathcal{H} = \{t \in \mathbb{R}_+^k : \prod_{i=1}^{k} t_i = 1\}$ denote the portion of hyperbola of equation $\prod_{i=1}^{k} t_i = 1$ contained in the positive orthant and $T = \{t \in \mathbb{R}_+^k : \prod_{i=1}^{k} t_i \geq 1\}$ be the (convex) set delimited by $\mathcal{H}$.

LEMMA 1. *For any positive real numbers $f_1, f_2, \ldots, f_k$ we have:*

$$\min_{t \in \mathcal{H}} \left(\frac{1}{k} \sum_{i=1}^{k} f_i t_i \right)^k = \prod_{i=1}^{k} f_i, \tag{3}$$

and the minimum point t^* of (3) satisfies

$$t_1^* f_1 = t_2^* f_2 = \cdots = t_k^* f_k = \left(\prod_{i=1}^{k} f_i\right)^{\frac{1}{k}}. \tag{4}$$

Proof. It follows easily from Kuhn–Tucker conditions. Indeed, associating multiplier μ_0 with the constraint

$$\prod_{i=1}^{k} t_i = 1 \tag{5}$$

and multiplier μ_j with the non negativity constraint

$$t_j \geq 0, \tag{6}$$

for $j = 1, 2, \ldots, k$, the Karush–Kuhn–Tucker conditions lead to:

$$f_j \left(\frac{1}{k}\sum_{i=1}^{k} f_i t_i\right)^{k-1} - \mu_0 \prod_{i=1, i\neq j}^{k} t_i - \mu_j = 0 \tag{7}$$

$$\mu_j t_j = 0 \tag{8}$$

$$\mu_j \geq 0 \tag{9}$$

for all $j = 1, 2, \ldots, k$.

Note that since none of the non negativity constraints can be satisfied as equalities, all points satisfying constraints (5) and (6) are regular.

Multiplying equation (7) by t_j, and using (5) and (8), we obtain

$$t_j f_j \left(\frac{1}{k}\sum_{i=1}^{k} f_i t_i\right)^{k-1} = \mu_0 \prod_{i=1}^{k} t_i + \mu_j t_j = \mu_0 \quad j = 1, 2, \ldots, k. \tag{10}$$

Summing the k equalities of (10), we get

$$\mu_0 = \left(\frac{1}{k}\sum_{i=1}^{k} f_i t_i\right)^{k}. \tag{11}$$

Since the coefficients f_i and the variables t_i are positive, $\frac{1}{k}\sum_{i=1}^{k} f_i t_i$ cannot be equal to 0. Thus, substituting μ_0 by its value obtained in (11) in equation (10), it follows

$$t_j f_j = \frac{1}{k}\sum_{i=1}^{k} f_i t_i \quad j = 1, 2, \ldots, k. \tag{12}$$

Making the product of these k last equalities, we obtain

$$\prod_{i=1}^{k} f_i = \left(\frac{1}{k}\sum_{i=1}^{k} f_i t_i\right)^{k}$$

which, together with equalities (12), completes the proof. ■

For every $t \in \mathbb{R}^k_+$, define the function φ as:

$$\varphi(t) = \min_{x \in G} \left\{ f_0(x) + \left(\frac{1}{k} \sum_{i=1}^{k} t_i f_i(x) \right)^k \right\}. \tag{13}$$

Since the functions $f_i (i = 1, 2, \ldots, k)$ are convex, the positive linear combination $\frac{1}{k} \sum_{i=1}^{k} t_i f_i$ is also convex. Since $y \to y^k$ defines a nondecreasing and convex function on $[0, +\infty)$ and since f_0 is also convex, it follows that $f_0 + \left(\frac{1}{k} \sum_{i=1}^{k} t_i f_i \right)^k$ is convex. Thus the value of $\varphi(t)$ can be determined by solving a convex program.

Other properties of φ are given below.

LEMMA 2. *The function φ is quasiconcave, continuous, increasing and, if f_0 is the null function, homogeneous of degree k, on any compact set D of $\mathbb{R}^k_+$.*

Proof. For any fixed $x \in G, t \to \sum_{i=1}^{k} t_i f_i(x)$ is an affine function. On the other hand, the function $y \to y^k$ is nondecreasing for $y \in [0, +\infty)$, hence $\left(\sum_{i=1}^{k} t_i f_i(x) \right)^k$ is a quasiconcave function of t (see, e.g., Avriel *et al.* [2, p. 57 Proposition 3.2]). It follows that, for any fixed $x \in G$ and any $\gamma \in \mathbb{R}$ the set

$$C_x(\gamma) = \left\{ t \in \mathbb{R}^k_+ : \left(\frac{1}{k} \sum_{i=1}^{k} t_i f_i(x) \right)^k \geq \gamma - f_0(x) \right\}$$

is convex. We then deduce that for any $\gamma \in \mathbb{R}$ the set $\{t \in \mathbb{R}^k_+ : \varphi(t) \geq \gamma\} = \bigcap_{x \in G} C_x(\gamma)$ is convex. This proves the quasiconcavity of $\varphi(t)$.

Let $\psi(t, x) = f_0(x) + \left(\frac{1}{k} \sum_{i=1}^{k} t_i f_i(x) \right)^k$. Since the $f_i, i = 0, 1, \ldots, k$ are continuous and since G is compact, ψ is uniformly continuous over $D \times G$. Thus, for any fixed $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|(t', x') - (t'', x'')\| \leq \delta \Rightarrow \|\psi(t', x') - \psi(t'', x'')\| \leq \varepsilon.$$

Now let t' and t'' be such that $\|t' - t''\| \leq \delta$. Let $x' \in \arg\min \varphi(t')$ and $x'' \in \arg\min \varphi(t'')$ (these points exist since G is a non-empty compact set).

On one hand,

$$\varphi(t') - \varphi(t'') = \min_{x \in G} \psi(t', x) - \min_{x \in G} \psi(t'', x) \leq \psi(t', x'') - \psi(t'', x'') \leq \varepsilon$$

(the last inequality holds since $\|(t', x'') - (t'', x'')\| = \|t' - t''\| \leq \delta)$.

On the other hand,

$$\varphi(t') - \varphi(t'') \geq \psi(t', x') - \psi(t'', x') \geq -\varepsilon.$$

Thus $|\varphi(t') - \varphi(t'')| \leq \varepsilon$ which proves the continuity of φ. Now, let x_t^* be the optimal solution of the convex program corresponding to $\varphi(t)$. For any $t \geq t' \geq 0, t \neq t'$ we have

$$\varphi(t) = f_0(x_t^*) + \left(\frac{1}{k}\sum_{i=1}^{k} t_i f_i(x_t^*)\right)^k > f_0(x_t^*) + \left(\frac{1}{k}\sum_{i=1}^{k} t'_i f_i(x_t^*)\right)^k \geq \varphi(t'),$$

hence $\varphi(t)$ is increasing.

The homogeneity property is trivial. ■

THEOREM 3. *Under assumption (2), the problem $(GCMP)$ is equivalent to the quasiconcave minimization problem*

$$(QCM) \qquad \min_{t \in T} \varphi(t)$$

in the following sense: if t^* is an optimal solution of problem (QCM) and if x^* is a corresponding point of G with respect to (13), then x^* is an optimal solution of problem $(GCMP)$. Moreover, the following relations hold:

$$t_i^* = \frac{\left(\prod_{j=1}^{k} f_j(x^*)\right)^{\frac{1}{k}}}{f_i(x^*)}, i = 1, 2, \ldots, k \tag{14}$$

$$f(x^*) = \varphi(t^*). \tag{15}$$

Conversely, if x^* is an optimal solution of problem $(GCMP)$, the value t^* deduced from the relation (14) corresponds to an optimal solution of problem (QCM) and the relation (15) holds.

Proof. Let x^* and f^* respectively be the optimal solution and value of problem $(GCMP)$, and t^* and φ^* be the optimal solution and value of problem (QCM).

Let x_{t^*} be a point of G such that $\varphi(t^*) = f_0(x_{t^*}) + \left(\frac{1}{k}\sum_{i=1}^{k} t_i^* f_i(x_{t^*})\right)^k$ (such a point exists by definition of φ). Then, by using Lemma 1,

$$\begin{aligned}\varphi^* = \varphi(t^*) \; &= \; f_0(x_{t^*}) + \left(\frac{1}{k}\sum_{i=1}^{k} t_i^* f_i(x_{t^*})\right)^k \\ &\geq \; f_0(x_{t^*}) + \min_{t \in \mathcal{H}} \left(\frac{1}{k}\sum_{i=1}^{k} t_i f_i(x_{t^*})\right)^k = f(x_{t^*}) \geq f^*. \end{aligned} \tag{16}$$

Now, let t_{x^*} be the point obtained from x^* by (14). Then

$$\begin{aligned}\varphi^* \; &\leq \; \varphi(t_{x^*}) = \min_{x \in G}\left\{ f_0(x) + \left(\prod_{i=1}^{k} f_i(x^*)\right) \left(\frac{1}{k}\sum_{i=1}^{k} \frac{f_i(x)}{f_i(x^*)}\right)^k \right\} \\ &\leq \; f(x^*) = f^* \end{aligned} \tag{17}$$

where the last inequality is obtained by setting x to x^*. From (16) and (17), it follows that $f^* = \varphi^*$.

Thus $f_0(x_{t^*}) + \left(\frac{1}{k}\sum_{i=1}^{k} t_i^* f_i(x_{t^*})\right)^k = f(x_{t^*}) = f^*$, which shows that x_{t^*} is an optimal solution of problem $(GCMP)$ and that (t^*, x_{t^*}) satisfies (14) (using again Lemma 1). Similarly, $\varphi(t_{x^*}) = \varphi^*$ which shows that t_{x^*} is an optimal solution of problem (QCM).

It follows that minimizing f over G is equivalent to minimizing φ over $\mathcal{H}$.

Since φ is increasing (Lemma 2), we deduce that the minimum of φ over T belongs to its boundary $\mathcal{H}$. Since T is a convex set, $(GCMP)$ is then equivalent to the quasiconcave minimization of $\varphi(t)$ over T. ■

COROLLARY 4. *Let* $\underline{f}_i > 0$ *and* $\overline{f}_i$ *be a lower and upper bound respectively of function* f_i *over* G *for* $i = 1, 2, \ldots, k$. *Let* t^* *be an optimal solution of problem* (QCM). *We have*

$$\underline{t}_i \le t_i^* \le \overline{t}_i \quad i = 1, 2, \ldots, k \tag{18}$$

where

$$\underline{t}_i = \frac{\left(\prod_{j\neq i} \underline{f}_j\right)^{\frac{1}{k}}}{(\overline{f}_i)^{1-\frac{1}{k}}} \quad \textit{and} \quad \overline{t}_i = \frac{\left(\prod_{j\neq i} \overline{f}_j\right)^{\frac{1}{k}}}{(\underline{f}_i)^{1-\frac{1}{k}}} \quad i = 1, 2, \ldots, k.$$

Note that if $\underline{t}_i = \overline{t}_i$ for some i and $k \ge 2$, the functions $f_j (j = 1, 2, \ldots, k)$ are constant over G, thus problem $(GCMP)$ can be reduced to the convex program $\min_{x\in G} f_0(x)$. Indeed,

$$\frac{\underline{t}_i}{\overline{t}_i} = \left(\frac{\left(\prod_{j\neq i} \underline{f}_j\right)^{\frac{1}{k}}}{(\overline{f}_i)^{1-\frac{1}{k}}}\right)\left(\frac{(\underline{f}_i)^{1-\frac{1}{k}}}{\left(\prod_{j\neq i}\overline{f}_j\right)^{\frac{1}{k}}}\right) = \left(\frac{\underline{f}_i}{\overline{f}_i}\right)^{1-\frac{2}{k}} \prod_{j=1}^{k}\left(\frac{\underline{f}_j}{\overline{f}_j}\right)^{\frac{1}{k}} \le 1$$

with the equality holding if and only if each term of the product is equal to 1, i.e., if $\underline{f}_j = \overline{f}_j$ for $j = 1, 2, \ldots, k$.

From now on, we assume that $0 < \underline{t}_i < \overline{t}_i$ for all $i \in \{1, 2, \ldots, k\}$.

3. Bounding and Branching Operations

In order to be able to design a conical branch-and-bound algorithm to solve problem (QCM), i.e., $\min_{t\in T}\varphi(t)$, we study below its various features: the initialization which includes the definition of a cone containing at least one optimal solution

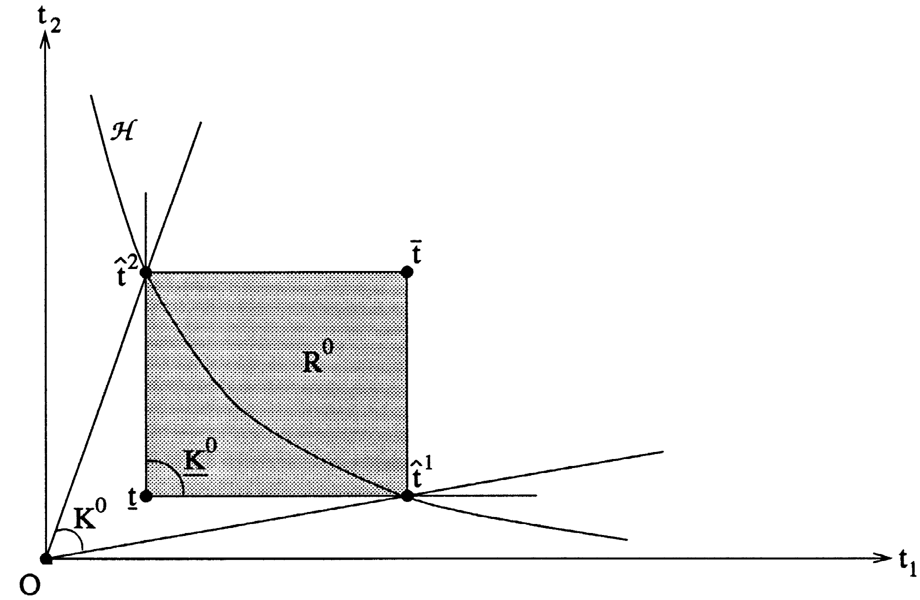

Figure 1. Construction of the initial cone $(k = 2)$.

(Section 3.1); the rules of the subdivision (Section 3.2); the computation of lower (Section 3.3) and upper bounds (Section 3.4).

3.1. Construction of an Initial Cone

Optimal solutions t^* of (QCM) are contained in the k-dimensional rectangle

$$R^0 = \{t \in \mathbb{R}^k : \underline{t} \leq t \leq \bar{t}\} \tag{19}$$

where $\underline{t} = (\underline{t}_1, \underline{t}_2, \ldots, \underline{t}_k)$ and $\bar{t} = (\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_k)$ are as defined in Corollary 4.

Let $\underline{K}^0$ be a first cone vertexed at $\underline{t}$ with edges defined by the k adjacent extreme points of $\underline{t}$ in R^0. Let $\hat{t}^j$ be the intersection of the j^{th} edge of $\underline{K}^0$ with the hyperbola $\mathcal{H}$ for $j = 1, 2, \ldots, k$. Such points always exist since on one hand $\underline{t}_i > 0$ for all $i \in \{1, 2, \ldots, k\}$ and on the other hand $\prod_{i=1}^k \underline{t}_i = \prod_{i=1}^k (\underline{f}_i / \overline{f}_i)^{1-\frac{1}{k}} < 1$ for $k \geq 2$.

We next consider K^0, the cone vertexed at O with edges $(O\hat{t}^j), j = 1, 2, \ldots, k$; see Figure 1 for an illustration when $k = 2$.

Note that for $k \geq 3$, the hyperrectangle R^0 is not always included in the cone K^0. Indeed, consider the following example with $k = 3$. Assume that $\underline{f}_i = \sqrt{2}^3 (i = 1, 2, 3)$ and $\overline{f}_i = \sqrt{3}^3 (i = 1, 2, 3)$. Then $\underline{t}_i = \frac{2}{3}$ and $\bar{t}_i = \frac{3}{2}$ for $i = 1, 2, 3$, and $1/\prod_{i=1}^k \underline{t}_i = \frac{27}{8}$. It follows that

$$\hat{t}^1 = \left(\frac{9}{4}, \frac{2}{3}, \frac{2}{3}\right) \quad \hat{t}^2 = \left(\frac{2}{3}, \frac{9}{4}, \frac{2}{3}\right) \quad \hat{t}^3 = \left(\frac{2}{3}, \frac{2}{3}, \frac{9}{4}\right).$$

Consider now the vertex $t' = (\underline{t}_1, \overline{t}_2, \overline{t}_3) = (\frac{2}{3}, \frac{3}{2}, \frac{3}{2})$ of R^0. It is easy to check that

$$t' = \frac{2}{817}\left(-4\hat{t}^1 + 211\hat{t}^2 + 211\hat{t}^3\right)$$

which shows that t' lies outside K^0.

PROPOSITION 5. *The cone K^0 contains the set of optimal solutions of problem (QCM).*

Proof. We first show that $\underline{t}$ belongs to K^0. By definition, the points $\hat{t}^j$ lie both on the j^{th} edge of $\underline{K}_0$, i.e., can be written

$$\hat{t}^j = \underline{t} + \lambda_j e^j \tag{20}$$

where e^j is the j^{th} unit vector and $\lambda_j \geq 0$, and on the hyperbola $\mathcal{H}$, i.e., satisfy

$$\prod_{i=1}^{k} \hat{t}_i^j = \frac{\underline{t}_j + \lambda_j}{\underline{t}_j} \prod_{i=1}^{k} \underline{t}_i = 1.$$

Setting $\tau = \prod_{i=1}^k \underline{t}_i = \prod_{i=1}^k (\underline{f}_i / \overline{f}_i)^{1-\frac{1}{k}}$, we get

$$\lambda_j = \left(\frac{1}{\tau} - 1\right) \underline{t}_j \quad j = 1, 2, \ldots, k. \tag{21}$$

Combining equations (20) and (21), we deduce:

$$\underline{t} = \frac{1}{k + \frac{1}{\tau} - 1} \sum_{j=1}^{k} \hat{t}^j \tag{22}$$

i.e., that t belongs to the cone K^0 (as $\tau < 1$).

As K^0 is convex and contains the points $\hat{t}^1, \hat{t}^2, \ldots, \hat{t}^k$ (by definition) and $\underline{t}$, it also contains the simplex $\underline{S}$ induced by these points.

From Corollary 4, all optimal solutions of problem (QCM) lie on $\underline{K}^0 \cap \mathcal{H}$. It therefore remains to show that $\underline{K}^0 \cap \mathcal{H} \subseteq \underline{S}$.

Let $t \in \underline{K}^0 \cap \mathcal{H}$. It satisfies the following system of equations:

$$\begin{cases} \prod_{i=1}^k t_i = 1 \\ t = \underline{t} + \displaystyle\sum_{j=1}^{k} \alpha_j (\hat{t}^j - \underline{t}) \end{cases}$$

with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \geq 0$.

By using (20) and (21), it follows that

$$t = \sum_{j=1}^{k} \left(1 + \alpha_j \left(\frac{1}{\tau} - 1\right)\right) \underline{t}_j e^j.$$

Since t belongs to the hyperbola $\mathcal{H}$, we have then

$$1 = \prod_{i=1}^{k} t_i = \tau \prod_{j=1}^{k} \left(1 + \alpha_j \left(\frac{1}{\tau} - 1\right)\right) \geq \tau \left(1 + \sum_{j=1}^{k} \alpha_j \left(\frac{1}{\tau} - 1\right)\right)$$

which shows that $\sum_{j=1}^{k} \alpha_j \leq 1$ (since $\frac{1}{\tau} - 1 > 0$). Thus $t = (1 - \sum_{j=1}^{k} \alpha_j)\underline{t} + \sum_{j=1}^{k} \alpha_j \hat{t}^j$, i.e., can be expressed as a convex combination of $\underline{t}$ and $\hat{t}^j (j = 1, 2, \ldots, k)$. This completes the proof. ■

The construction of K^0 requires the knowledge of $\underline{t}$. However, in order to be able to compute $\underline{t}$, we need lower and upper bound on each function of the product in f.

Positive lower bounds $\underline{f}_i$ can be easily obtained through the solution of the following convex problems

$$\min_{x \in G} f_i(x)$$

for $i = 1, 2, \ldots, k$.

The computation of upper bounds requires more effort. If the function f_i is linear, an upper bound $\overline{f}_i$ can be obtained by solving the following convex problem

$$\max_{x \in G} f_i(x).$$

If the function f_i is not linear, let Σ be a simplex containing G. Without loss of generality, we can assume that G is contained in the positive orthant. Then a simplex containing G can be defined as $\Sigma = \{x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i \leq \overline{b}; x_i \geq 0, i = 1, 2, \ldots, n\}$ where $\overline{b}$ is the optimal value of the convex program

$$\max_{x \in G} \sum_{i=1}^{k} x_i.$$

Then we can derive an upper bound $\overline{f}_i$ by solving the convex maximization problem

$$\max_{x \in \Sigma} f_i(x).$$

Since Σ is a polytope whose vertices can easily be computed, this last optimization problem can be easily solved.

3.2. Subdivision

We propose to consider bisection subdivisions. We recall below the principles of such subdivisions. The reader is referred to Horst and Tuy [7] and Tuy [28] for more details.

Let H^0 be a hyperplane intersecting each edge of K^0, e.g., the hyperplane of equation $\sum_{i=1}^{k} t_i = 1$. At a current iteration, let $K \subseteq K^0$ be a cone vertexed at the origin O and with k independent edges. let $U = K \cap H^0 = \text{conv}\{u^1, u^2, \ldots, u^k\}$ be the section of K by the hyperplane H^0: U is called the *base* of K. Let w be an arbitrary point of U such that

$$w = \sum_{i=1}^{k} \lambda_i u^i, \quad \sum_{i=1}^{k} \lambda_i = 1, \quad \lambda_i \geq 0 \quad (i = 1, 2, \ldots, k). \tag{23}$$

Let $I = \{i : \lambda_i > 0\}$. For each $i \in I$, define U_i as the simplex of vertices $u^1, \ldots, u^{i-1}, w, u^{i+1}, \ldots, u^k$. It is easy to verify that the set of simplices $\{U_i : i \in I\}$ forms a partition of the simplex U, it is called a *simplicial subdivision*. Let $\delta(U)$ denote the diameter of the simplex U, i.e., the length of its longest edge. If w belongs to a longest edge of U, i.e., if $w = \alpha u^p + (1 - \alpha) u^q$ with $0 < \alpha \leq \frac{1}{2}$ and $\|u^p - u^q\| = \delta(U)$, then the partition is called a *bisection* of ratio α.

DEFINITION 6 (see Horst and Tuy [7, p. 135]). A simplicial subdivision is exhaustive if any infinite sequence of nested simplices U^h satisfies $\lim_{h \to +\infty} \delta(U^h) = 0$.

THEOREM 7 (see, e.g., Tuy [28, p. 21]). *A subdivision process consisting exclusively of bisections of ratio* $0 < \alpha \leq \frac{1}{2}$*, for some fixed* α*, is exhaustive.*

Obviously, any partition of U induces a partition of K. If the simplex U is bisected, we say that the cone K is bisected. If the subdivision of U is exhaustive, we say that the subdivision of K is exhaustive.

3.3. Lower Bounds

We propose two ways to compute lower bounds: the first one applies to the general case and requires k evaluations of φ, i.e., the solution of k convex programs in $\mathbb{R}^n$; the second one requires only one evaluation of φ but does not apply for $k \geq 3$.

3.3.1. *Cutting Plane Method*

The computation of the first lower bound is based on the following result:

PROPOSITION 8. *Let* $K \subset \mathbb{R}^k_+$ *be a cone vertexed at* O*. Let* H *be a hyperplane separating* O *from* $K \cap T$ *and let* s^j $(j = 1, 2, \ldots, k)$ *be the intersection points of the edges of* K *with* H*. Then*

$$\min\{\varphi(s^1), \varphi(s^2), \ldots, \varphi(s^k)\}$$

is a lower bound of φ *over* $K \cap T$*.*

Proof. Let Q be the polyhedron defined as the intersection of K with the half-space delimited by H and not containing O. Clearly Q contains $K \cap T$. Since φ is increasing, its minimum over Q is attained at $\text{conv}\{s^1, s^2, \ldots, s^k\}$. As φ is quasiconcave, it follows that the minimum of φ over Q is equal to $\min\{\varphi(s^1), \varphi(s^2), \ldots, \varphi(s^k)\}$. ∎

Let $\hat{t}$ be a point of $\mathcal{H}$. We denote by $H_{\hat{t}}$ the hyperplane tangent to the hyperbola $\mathcal{H}$ at point $\hat{t}$. It can easily be checked that

$$H_{\hat{t}} = \left\{ t \in \mathbb{R}^k : \sum_{i=1}^{k} \frac{t_i}{\hat{t}_i} = k \right\}.$$

Clearly, $H_{\hat{t}}$ separates O from $K \cap T$, thus yields a lower bound by Proposition 8. For convenience, let us define $a = (\frac{1}{\hat{t}_1}, \frac{1}{\hat{t}_2}, \ldots, \frac{1}{\hat{t}_k})$. Then $H_{\hat{t}} = \{t \in \mathbb{R}^k : at = k\}$. Since the minimum of φ over T is attained at a point of $\mathcal{H}$, a good criteria for selecting the hyperplane $H_{\hat{t}}$ may be to minimize the volume of the set $\mathcal{S} = K \cap \{t : at \geq k, \prod_{i=1}^{k} t_i \leq 1\}$.

PROPOSITION 9. *A necessary condition for the hyperplane* $H = \{t \in \mathbb{R}^k : at = k\}$ *to minimize the volume of* $\mathcal{S}$ *is*

$$a_i = \frac{1}{\displaystyle\sum_{j=1}^{k} \frac{\hat{t}_i^j}{a\hat{t}^j}}, \quad i = 1, 2, \ldots, k. \tag{24}$$

Proof. Denote by $\hat{H} = \{t \in \mathbb{R}^k : \hat{a}t = 1\}$ the hyperplane passing through the points $\hat{t}^j, j = 1, 2, \ldots, k$. Then $\mathcal{S} = \mathcal{S}_1 \backslash (\mathcal{S}_2 \cup \mathcal{S}_3)$ where

$$\begin{aligned}
\mathcal{S}_1 &= K \cap \{t \in \mathbb{R}^k : \hat{a}t \leq 1\}, \\
\mathcal{S}_2 &= K \cap \{t \in \mathbb{R}^k : at \leq k\}, \\
\mathcal{S}_3 &= K \cap \left\{ t \in \mathbb{R}^k : \hat{a}t \leq 1, \prod_{i=1}^{k} t_i \geq 1 \right\}.
\end{aligned}$$

Since only the set $\mathcal{S}_2$ depends upon the hyperplane H, minimizing the volume of $\mathcal{S}$ is equivalent to maximizing the volume of $\mathcal{S}_2$. The vertices of this simplex are O and $s^j = \lambda_j \hat{t}^j$ with $\lambda_j = \frac{k}{a\hat{t}^j}$ for $j = 1, 2, \ldots, k$. Thus

$$\begin{aligned}
Vol(\mathcal{S}_2) &= \nu_k \left| det \begin{pmatrix} 0 & \lambda_1 \hat{t}^1 & \lambda_2 \hat{t}^2 & \cdots & \lambda_k \hat{t}^k \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \right| \\
&= \nu_k |det(\lambda_1 \hat{t}^1 \quad \lambda_2 \hat{t}^2 \quad \cdots \quad \lambda_k \hat{t}^k)| \\
&= \nu_k \left(\prod_{i=1}^{k} \lambda_i \right) |det(\hat{t}^1 \hat{t}^2 \cdots \hat{t}^k)|
\end{aligned}$$

where ν_k is a constant for fixed k. Since $|det(\hat{t}^1, \hat{t}^2, \ldots, \hat{t}^k)|$ does not depend upon a, the hyperplane H minimizing the volume of $\mathcal{S}$ is obtained by solving the following problem

$$(VP) \qquad \min v(a) = \prod_{j=1}^{k} (a\hat{t}^j)$$

$$\text{s.t.} \begin{cases} \prod_{i=1}^{k} a_i = 1 & (25) \\ a \geq 0. & (26) \end{cases}$$

Noting that no a_i can be null, the Karush–Kuhn–Tucker conditions lead to

$$\sum_{j=1}^{k} \hat{t}_\ell^j \left(\frac{v(a)}{a\hat{t}^j} \right) - \mu \left(\frac{\prod_{i=1}^{k} a_i}{a_\ell} \right) = 0, \qquad \ell = 1, 2, \ldots, k \tag{27}$$

$$\mu \left(\prod_{i=1}^{k} a_i - 1 \right) = 0. \tag{28}$$

Using (25), relation (27) gives

$$\frac{\mu}{v(a)} = a_\ell \sum_{j=1}^{k} \frac{\hat{t}_\ell^j}{a\hat{t}^j}, \quad \ell = 1, 2, \ldots, k.$$

After summation, we obtain $\frac{\mu}{v(a)} = 1$, which concludes the proof. ∎

Note that the hyperplane that minimizes the volume of $\mathcal{S}$ is tangent to $\mathcal{H}$ at a point of K. Indeed relation (24) can be written

$$\hat{t} = \sum_{j=1}^{k} \frac{\hat{t}^j}{a\hat{t}^j}$$

where the $a\hat{t}^j, j = 1, 2, \ldots, k$ are positive since the components of a and $\hat{t}^j$ are positive. For $k = 2$, such a hyperplane can be more precisely characterized (see Figure 2 for an illustration).

COROLLARY 10. *For $k = 2$, the hyperplane that minimizes the volume of $\mathcal{S}$ is the hyperplane tangent to $\mathcal{H}$ at point $\hat{t} = (\sqrt{\hat{t}_1^1 \hat{t}_1^2}, \sqrt{\hat{t}_2^1 \hat{t}_2^2})$. Moreover, if $\bar{\hat{t}}$ denotes the barycentre of $\hat{t}^1$ and $\hat{t}^2$, then $\hat{t}$ is the intersection of $O\bar{\hat{t}}$ with $\mathcal{H}$.*

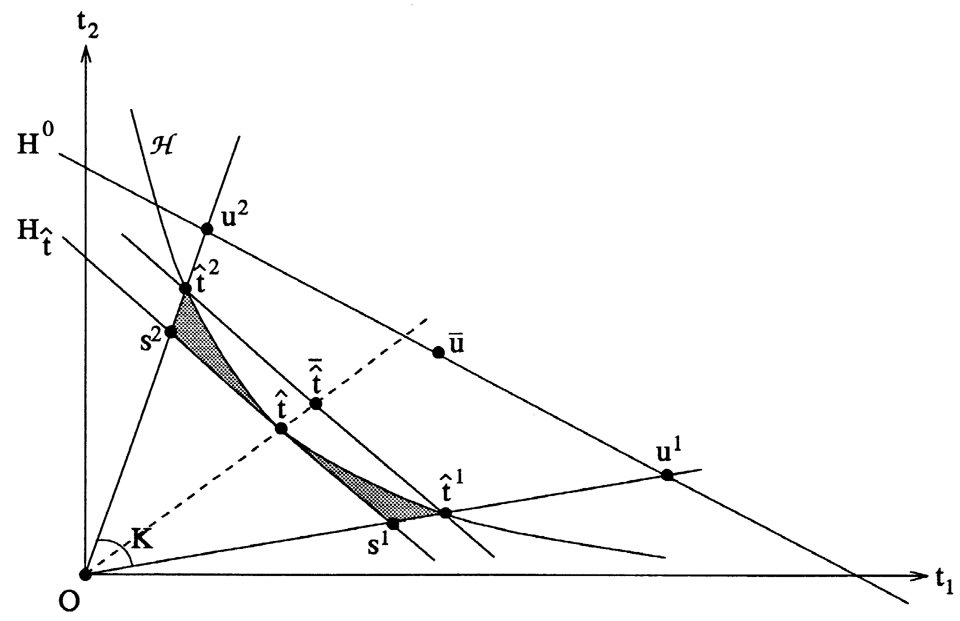

Figure 2. Lower bound computing (method 1).

Proof. Using relation (24) of Proposition 9, we have

$$\begin{aligned}\frac{1}{a_1} &= \frac{\hat{t}_1^1}{a_1\hat{t}_1^1 + a_2\hat{t}_2^1} + \frac{\hat{t}_1^2}{a_1\hat{t}_1^2 + a_2\hat{t}_2^2} \\ &= \frac{\hat{t}_1^1(a_1\hat{t}_1^2 + a_2\hat{t}_2^2) + \hat{t}_1^2(a_1\hat{t}_1^1 + a_2\hat{t}_2^1)}{(a_1\hat{t}_1^1 + a_2\hat{t}_2^1)(a_1\hat{t}_1^2 + a_2\hat{t}_2^2)}.\end{aligned}$$

After simplification, we obtain $(\hat{t}_1^1\hat{t}_1^2)a_1^2 = (\hat{t}_2^1\hat{t}_2^2)a_2^2$, i.e., $(\hat{t}_1^1\hat{t}_1^2)\hat{t}_2^2 = (\hat{t}_2^1\hat{t}_2^2)\hat{t}_1^2$. Since $\hat{t}_1\hat{t}_2 = 1 = \hat{t}_1^1\hat{t}_2^1 = \hat{t}_1^2\hat{t}_2^2$, it follows that the only positive solution is $\hat{t}_1 = \sqrt{\hat{t}_1^1\hat{t}_1^2}$ and $\hat{t}_2 = \sqrt{\hat{t}_2^1\hat{t}_2^2}$. Using the equalities $\hat{t}_1^1\hat{t}_2^1 = 1 = \hat{t}_1^2\hat{t}_2^2$, it is then easy to verify that $\hat{t} = (\frac{2\sqrt{\hat{t}_1^1\hat{t}_2^1}}{\hat{t}_1^1+\hat{t}_1^2})\bar{\hat{t}}$. ∎

Considering this last result, we define three variants for the computation of a lower bound that differ by the choice of $\hat{t}$:

(a) $\hat{t}$ is defined by

$$\hat{t}_i = \sqrt[k]{\prod_{j=1}^{k} \hat{t}_i^j}, \quad i = 1, 2, \ldots, k.$$

(b) $\hat{t}$ is the intersection with $\mathcal{H}$ of $O\bar{\hat{t}}$, where $\bar{\hat{t}}$ is the barycentre $\frac{1}{k}\sum_{j=1}^{k} \hat{t}^j$ of the points $\hat{t}^j$.

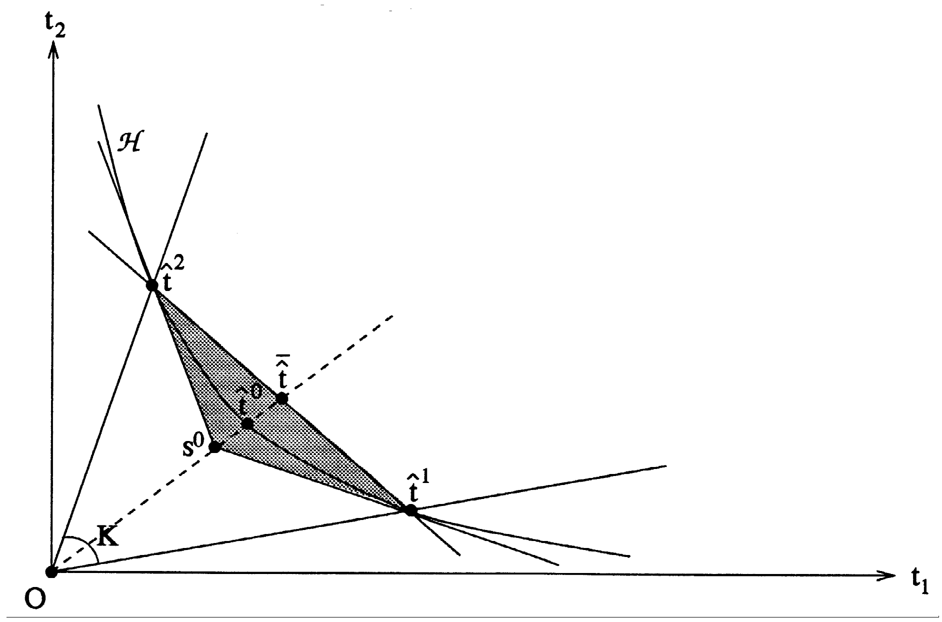

Figure 3. Lower bound computing (method 2).

(c) $\hat{t}$ is the intersection with $\mathcal{H}$ of $O\overline{u}$, where $\overline{u}$ is the barycentre of the points u^j, intersection of $O\hat{t}^j$ with the hyperplane H^0 defined in Section 3.2.

Note that for $k \geq 3$, the corresponding hyperplanes will not anymore minimize the volume of $\mathcal{S}$ since $a = (\frac{1}{\hat{t}_1}, \frac{1}{\hat{t}_2}, \ldots, \frac{1}{\hat{t}_k})$ does not in general satisfy (24).

3.3.2. *Simplicial Method*

The computation of the second lower bound is valid only when $k = 2$. It exploits a particular simplex (see Figure 3 for an illustration):

PROPOSITION 11. *Let $K \subset \mathbb{R}^2_+$ be a cone origined at O. Assume that its edges intersect the hyperbola $\mathcal{H}$ at $\hat{t}^1$ and $\hat{t}^2$ respectively. Let $H_{\hat{t}^1}$ and $H_{\hat{t}^2}$ be the hyperplanes tangent to $\mathcal{H}$ at $\hat{t}^1$ and $\hat{t}^2$ respectively. Then these hyperplanes are intersecting at an unique point $s^0 \in K$, and a lower bound of φ over $K \cap \mathcal{H}$ is*

$$\min\{\varphi(\hat{t}^1), \varphi(\hat{t}^2), \varphi(s^0)\}.$$

Proof. Let $\hat{H}$ be the hyperplane passing through $\hat{t}^1$ and $\hat{t}^2$. By convexity of T, the half-space delimited by H and containing O contains $K \cap \mathcal{H}$. Since $H_{\hat{t}^1}$ and $H_{\hat{t}^2}$ are supporting hyperplanes, $K \cap \mathcal{H}$ is therefore included in the simplex S defined by the hyperplanes H, $H_{\hat{t}^1}$ and $H_{\hat{t}^2}$. It follows that a lower bound of φ over $K \cap \mathcal{H}$ is $\min_{t \in S} \varphi(t)$. As φ is quasiconcave, it is equal to the minimum of φ over the extreme points of S.

These extreme points are $\hat{t}^1$, $\hat{t}^2$ and the intersection point s^0 of the hyperplanes $H_{\hat{t}^1}$ and $H_{\hat{t}^2}$. If $\hat{t}^1 = (\alpha, \frac{1}{\alpha})$ and $\hat{t}^2 = (\beta, \frac{1}{\beta})$, the equations of the hyperplanes $H_{\hat{t}^1}$ and $H_{\hat{t}^2}$ are:

$$\frac{s_1}{\alpha} + \alpha s_2 = 2$$

$$\frac{s_1}{\beta} + \beta s_2 = 2.$$

This last system has an unique solution

$$s^0 = \left(\frac{2\alpha\beta}{\alpha+\beta}, \frac{2}{\alpha+\beta}\right) = \frac{\alpha\beta}{(\alpha+\beta)^2}(\hat{t}^1 + \hat{t}^2)$$

which clearly belongs to K. ■

REMARK 12. Let $\hat{t}^0$ be the intersection point of Os^0 with the hyperbola $\mathcal{H}$. Then $s^0 = (\frac{2\sqrt{\alpha\beta}}{\alpha+\beta})\hat{t}^0$. On the other hand, with the above notation and with the hyperplane of Corollary 10, it can be checked that the points s^j of Proposition 8 satisfy $s^j = (\frac{2\sqrt{\alpha\beta}}{\alpha+\beta})\hat{t}^j$ for $j = 1, 2$. Therefore, for $k = 2$, both the cutting plane lower bound method (described in Section 3.3.1) and the current method evaluate φ at points that are in the same proportion with respect to the hyperbola. Since the first method takes the worst value of two such points while the second considers only one point, we can expect that for $k = 2$ the latter is better than the former.

REMARK 13. As described above, the second lower bound method requires $k+1$ evaluations of φ. However, the algorithm can be adapted in such a way that the computations of $\varphi(\hat{t}^1)$ and $\varphi(\hat{t}^2)$ are made when computing the upper bound (see Section 3.4 and 4.2).

Unfortunately, the computation of this second lower bound does not extend easily to the case $k \geq 3$. A natural generalization would be to consider the simplex defined by the hyperplanes tangent to $\mathcal{H}$ at the points $\hat{t}^j, j = 1, 2, \ldots, k$ and by the hyperplane passing through these points. However, as shown by the following example, some extreme points of this simplex can lie outside the positive orthant, i.e., in a region in which the objective function φ is not defined.

EXAMPLE 14. Assume that $k > 2$ and let a and b be two positive reals satisfying

$$a < \sqrt[k]{\frac{k-2}{k-1}} \tag{29}$$

$$b = \frac{1}{a^{k-1}}. \tag{30}$$

Let $K \subset \mathbb{R}^k$ be the cone defined by $\hat{t}^j, j = 1, 2, \ldots, k$ where

$$\hat{t}_i^j = \begin{cases} a & \text{if } i \neq j \\ b & \text{if } i = j. \end{cases}$$

Note that conditions (29) and (30) imply that $a \neq b$ (actually we have $a < b$) which shows that K is nondegenerated.

The equation of the hyperplane passing through the $\hat{t}^j$ is

$$\sum_{i=1}^{k} t_i = (k-1)a + b, \tag{31}$$

while the equation of the hyperplane tangent to $\mathcal{H}$ at point $\hat{t}^j$ is

$$\sum_{i=1, i\neq j}^{k} \frac{t_i}{a} + \frac{t_j}{b} = k. \tag{32}$$

The extreme point s^p corresponding to the system (31) and (32, $j \neq p$) is thus defined by

$$s_i^p = \begin{cases} b & \text{if } i \neq p \\ (k-1)a - (k-1)b & \text{if } i = p. \end{cases}$$

Since $(k-1)a - (k-2)b = \frac{(k-1)a^k - (k-2)}{a^{k-1}}$, it follows from assumption (29) that $s_p^p < 0$, thus the k extreme points $s^p, p = 1, 2, \ldots, k$ are not in the positive orthant.

This example shows that we must add additional constraints in order to keep the extreme points of the outer-approximating polytope in the positive orthant. In order to obtain a tight approximation we can for example add the constraints defining the cone K. The above continuation of Example 14 shows then that the resulting polytope may have an exponential number of extreme points (we would have obtained a similar result by adding the positivity constraints instead of the constraints defining K).

EXAMPLE 14 (*continuation).* Recall that the constraints defining our simplex are

$$\sum_{i=1}^{k} t_i + \left(\frac{a}{b} - 1\right) t_j \geq ka, \quad j = 1, 2, \ldots, k \tag{33}$$

$$\sum_{i=1}^{k} t_i \leq (k-1)a + b. \tag{34}$$

Let us complete it with the constraints defining K. It is easy to verify that these constraints are

$$a \sum_{i=1}^{k} t_i - ((k-1)a + b)t_j \leq 0, \quad j = 1, 2, \ldots, k. \tag{35}$$

Let J_1, J_2 be a partition of $\{1, 2, \ldots, k\}$ with $J_1 \neq \emptyset$. We claim that the point $t = (t_1, t_2, \ldots, t_k)$ with

$$t_j = \begin{cases} \dfrac{kab(|J_1|a + b - a)}{kab|J_1| + (b-a)^2(|J_1| - 1)} & \text{if } j \in J_1 \\ \dfrac{ka^2b|J_1|}{kab|J_1| + (b-a)^2(|J_1| - 1)}, & \text{if } j \in J_2 \end{cases} \tag{36}$$

is an extreme point of the polytope defined by constraints (33), (34) and (35). Indeed, we have

$$\begin{aligned} \sum_{j=1}^{k} t_j &= \frac{kab|J_1|(|J_1|a + b - a + (k - |J_1|)a)}{kab|J_1| + (b-a)^2(|J_1| - 1)} \\ &= \frac{kab|J_1|((k-1)a + b)}{kab|J_1| + (b-a)^2(|J_1| - 1)}. \end{aligned} \tag{37}$$

Clearly since $|J_1| \geq 1$, $\frac{kab|J_1|}{kab|J_1| + (b-a)^2(|J_1|-1)} \leq 1$, which shows that constraint (34) is satisfied.

For $j \in J_1$, the left hand of (33) is

$$\frac{kab|J_1|(ka + b - a) + ka(a-b)(|J_1|a + b - a)}{kab|J_1| + (b-a)^2(|J_1| - 1)} = ka$$

thus the constraint is satisfied at equality.

For $j \in J_2$, the left hand of (33) is

$$\frac{kab|J_1|(ka+b-a)+ka^2|J_1|(a-b)}{kab|J_1|+(b-a)^2(|J_1|-1)} = \frac{ka[kab|J_1|+(b-a)^2|J_1|]}{kab|J_1|+(b-a)^2(|J_1|-1)} > ka.$$

For $j \in J_1$, the left hand of (35) is

$$\frac{kab[ka+b-a][a|J_1| - (|J_1|a+b-a)]}{kab|J_1|+(b-a)^2(|J_1|-1)} = \frac{kab(a-b)[ka+b-a]}{kab|J_1|+(b-a)^2(|J_1|-1)} < 0$$

since $a < b$. Finally, it is clear that for $j \in J_2$, (35) is satisfied at equality.

Since there are $2^k - 1$ nonempty distinct subsets of $\{1, 2, \ldots, k\}$, the polytope defined by (33)–(35) has at least $2^k - 1$ extreme points.

This example shows that the generalization of the simplicial bound to $k \geq 3$ implies the enumeration of a number of extreme points that can growth exponentially with k. Since for each of these points, we have to solve a convex program in $\mathbb{R}^n$, it seems unlikely that this method could be efficient for $k \geq 3$.

3.4. Upper Bounds

An upper bound of φ over $K \cap \mathcal{H}$ is obtained by evaluating φ at a point w, which is a by-product of the computation of the lower bound.

If the cutting plane lower bound is used, the point w is the intersection of $O\hat{t}$ with the hyperbola $\mathcal{H}$ where $\hat{t}$ is as defined in Section 3.3.1.

If the simplicial lower bound is used, the point w is defined as the intersection of Os^0 with $\mathcal{H}$.

4. A Conical Algorithm

4.1. General Case

We present below a conical algorithm, called SOLQCM, which provides an ε-optimal solution for any parameter $\varepsilon \geq 0$. If $\varepsilon = 0$, an optimal solution for problem (GCMP) can be easily deduced using Theorem 3. Otherwise, a minor modification, discussed at the end of this section, must be made in order to be able to deduce an ε-optimal solution for (GCMP) from an ε-optimal solution for (QCM).

Algorithm SOLQCM

Step 0 **(initialization)**: select one of the two lower bound methods described in Section 3.3.

Set $\overline{\varphi}$ to $\tilde{f}$ where $\tilde{f}$ is the minimum of the optimal values $\tilde{f}_i$ of the problems $(P_i), i = 1, 2, \ldots, k$ as defined in Section 2.

Construct an initial cone K^0 as described in Section 3.1.

Compute the lower bound $\underline{\varphi}(K^0)$.

Set the list $\mathcal{L}$ of subproblems to $\{K^0\}$.

Step 1 **(subdivision)**: let $K^* \in \arg\min\{\underline{\varphi}(K) : K \in \mathcal{L}\}$.

Perform a bisection of K^*. Let $\mathcal{P}$ be the bipartition of K^*. Set $\mathcal{L} \leftarrow (\mathcal{L} \backslash \{K^*\}) \cup \mathcal{P}$.

Step 2 **(bounding)**: for each cone $K \in \mathcal{P}$, compute the lower bound $\underline{\varphi}(K)$. Let $w(K)$ be the point defined in Section 3.4.

If for some $K \in \mathcal{P}$, $\varphi(w(K)) < \overline{\varphi}$ then set $\overline{\varphi} \leftarrow \varphi(w(K))$ and $\overline{t} \leftarrow w(K)$.

Step 3 **(fathoming)**: delete every cone $K \in \mathcal{L}$ for which $\underline{\varphi}(K) \geq \overline{\varphi} - \varepsilon$. If $\mathcal{L} = \emptyset$ then terminate: $\overline{\varphi}$ is an ε-optimal solution of (QCM); otherwise return to Step 1.

THEOREM 15. *Algorithm SOLQCM is correct and can be infinite only if $\varepsilon = 0$. In such a case, any cluster point of the sequence $\overline{t}$ is an ε-optimal solution of problem (QCM).*

Proof. Let $\mathcal{L}'$ be the set of cones either in $\mathcal{L}$ or deleted at some iteration in Step 3. Clearly $\mathcal{L}'$ forms a partition of cone K^0 thus $\min\{\underline{\varphi}(K) : K \in \mathcal{L}'\} \leq \min\{\varphi(t) : t \in \mathcal{H}\} = \varphi^*$. If at some iteration $\mathcal{L} = \emptyset$ then $\overline{\varphi} - \varepsilon \leq \min\{\underline{\varphi}(K) : K \in \mathcal{L}'\} \leq \varphi^*$ which proves that $\overline{t}$ is an ε-optimal solution of problem (QCM).

Now assume that the algorithm is infinite. Since at each iteration a cone is subdivided into a finite number of subcones, it must generate an infinite sequence of nested cones K^h.

Let $\overline{\varphi}^h$ and $\overline{t}^h$ be the incumbent value and point respectively at iteration h. Since the sequence $\overline{\varphi}^h$ is decreasing and bounded by $\min_{t\in\mathcal{H}}\varphi(t)$, $\overline{\varphi}^h$ converges to a limit $\overline{\varphi}$ and consequently, since $K^0 \cap \mathcal{H}$ is a compact set, the sequence $\overline{t}^h$ has at least one cluster point $\overline{t}$.

Let $\hat{t}^{hj}, j = 1, 2, \ldots, k$ be the intersection points of the edges of K^h with the hyperbola $\mathcal{H}$. Since the subdivision process involves only bisections, it is exhaustive following Theorem 7. Therefore, the sequences $(\hat{t}^{hj})_h$ converge to a common limit $\hat{t}^*$.

If the cutting plane method is used to compute the lower bounds $\underline{\varphi}^h = \underline{\varphi}(K^h)$, let $\hat{t}^h$ be the point $\hat{t}$ at iteration h, H^h be the hyperplane $H_{\hat{t}^h}$ and $s^{hj}, j = 1, 2, \ldots, k$ be the intersection point of the edges of K^h with H^h. Clearly, $H^h \to H^*$ which is the hyperplane tangent to $\mathcal{H}$ at $\hat{t}^*$ and, for all $j, s^{hj} \to \hat{t}^*$. Moreover, $w^h \to \hat{t}^*$. By continuity of φ, we have

$$\begin{aligned}\min\{\varphi(t) : t \in \mathcal{H}\} \geq \lim_{h\to\infty} \underline{\varphi}^h &= \lim_{h\to+\infty} \min\{\varphi(s^{h1}), \varphi(s^{h2}), \ldots, \varphi(s^{hk})\}\\ &= \varphi(\hat{t}^*)\\ &= \lim_{h\to+\infty} \varphi(w^h) \geq \lim_{h\to+\infty} \varphi(\overline{t}^h) = \varphi(\overline{t}),\end{aligned}$$

thus $\overline{t}$ is an optimal solution of problem (QCM).

Similarly, if the simplicial lower bound method is used, let s^{h_0} be the point s^0 at iteration h. Since, for all $j, \hat{t}^{hj} \to \hat{t}^*$, we have $H_{\hat{t}^{hj}} \to H^*$ for all j where H^* is the hyperplane tangent to $\mathcal{H}$ at point $\hat{t}^*$. Since s^{h0} is on the hyperplanes $H_{\hat{t}^{hj}}, j = 1, 2, \ldots, k$ and in the cone K^h whose limit is the edge $\{O\hat{t}^*\}$, we have $s^{h0} \to \hat{t}^*$ and $w^h \to \hat{t}^*$. By continuity of φ, we have then

$$\begin{aligned}\min\{\varphi(t) : t \in \mathcal{H}\} \geq \lim_{h\to+\infty} \underline{\varphi}^h &= \lim_{h\to+\infty} \varphi(s^{h0}) = \varphi(\hat{t}^*)\\ &= \lim_{h\to+\infty} \varphi(w^h) \geq \lim_{h\to+\infty} \varphi(\overline{t}^h) = \varphi(\overline{t}),\end{aligned}$$

which shows again that $\overline{t}$ is an optimal solution of problem (QCM).

Since at Step 3, cones satisfying $\underline{\varphi}(K^h) \geq \overline{\varphi}^h - \varepsilon$ are deleted, we cannot have $\lim_{h\to+\infty} \underline{\varphi}(K^h) = \lim_{h\to+\infty} \overline{\varphi}^h$ for $\varepsilon > 0$. Thus the algorithm cannot be infinite if $\varepsilon > 0$. ■

If $\overline{t}$ is an optimal solution of problem (QCM), then by Theorem 3 the point $x_{\overline{t}}$ of G associated with $\overline{t}$ in the evaluation of $\varphi(\overline{t})$ is an optimal solution of problem $(GCMP)$. However, if $\overline{t}$ is an ε-optimal solution of problem (QCM), $\varphi(\overline{t})$ is still an ε-optimal value of problem $(GCMP)$ but $x_{\overline{t}}$ is not necessarily an ε-optimal point, i.e., a point whose value differs by less than ε from the optimal one.

To provide an ε-optimal solution of problem $(GCMP)$, the algorithm SOLQCM should be modified in the following way: Step 2, the last line should be replaced by:

> If for some $k \in \mathcal{P}, f(x_{w(K)}) < \overline{\varphi}$ then set $\overline{\varphi} \leftarrow f(x_{w(K)})$ and $\overline{x} \leftarrow x_{w(K)}$ where $x_{w(K)}$ is the point of G solution of the convex program corresponding to the evaluation of $\varphi(w(K))$.

In order to prove the finiteness of the resulting modified algorithm, note that in the proof of Theorem 15, $w^h \to \hat{t}^*$ which is an optimal solution of problem (QCM). Let x_{w^h} be the point of G associated to w^h in the evaluation of $\varphi(w^h)$: $x_{w^h} \to x_{\hat{t}^*}$. Thus $\varphi(t^*) = f(x_{\hat{t}^*}) = \lim_{h\to+\infty} f(x_{w^h}) \geq \lim_{h\to+\infty} f(\overline{x}^h) = f(\overline{x})$. The rest of the proof is similar to that of Theorem 15.

Note that the classical lower bound computation method using γ-extensions (see for example Horst and Tuy [7]) would not be practicable here, due to the difficulty of evaluating φ.

Another possible choice for the origin of the cones would have been to consider a vertex of the hyperrectangle R^0. But then we would have lost the property of homogeneity when $f_0 = 0$ (see Lemma 2), particularly useful in practice to save some computing time as discussed in the next section.

4.2. Special Cases

When f_0 is constant, or without loss of generality when $f_0 = 0$, some simplifications can be made.

First note that the convex program

$$\min_{x\in G}\left[\frac{1}{k}\sum_{i=1}^{k} t_i f_i(x)\right]^k$$

needed to evaluate φ at point t is equivalent to

$$\min_{x\in G}\sum_{i=1}^{k} t_i f_i(x) \tag{38}$$

since the function $y \to y^k$ is increasing on $[0, +\infty)$. In particular, if the functions $f_i, i = 1, 2, \ldots, k$ are linear and if G is a polytope, then the optimization problem (38) reduces to a linear program. Therefore it may be convenient to replace φ by its reduced form:

$$\varphi'(t) = \min_{x\in G}\sum_{i=1}^{k} t_i f_i(x). \tag{39}$$

Also recall that by Lemma 2, φ is homogeneous of degree k when f_0 is the null function. Therefore, the value of φ of an entire edge can easily be deduced from the knowledge of the value at a particular point of this edge. This is especially useful when the cutting plane lower bound is used since the points to be evaluated are on edges which usually belong to more than one cone.

If $k = 2$, we can also subdivide the cone K using the point $w(K)$: in dimension 2, this corresponds to a bisection, therefore the proof of Theorem 15 remains valid. This subdivision method is particularly interesting when the simplicial lower bound is used: in such a case, all edges, except those defining the initial cone intersect $\mathcal{H}$ at a point of the form $w(K)$, the value of which has been calculated at Step 2 in an attempt to improve the incumbent value. Hence when computing a simplicial lower bound the only point to evaluate is s^0 (see Section 3.3.2).

5. Computational Results

In this section, we present the results of computational experiments for two versions of algorithm SOLQCM: SOLQCM1 in which the first lower bound is used and SOLQCM2 in which the second lower bound is used.

We consider test problems similar to those used in the literature. There are of five types described in Table I.

Problems of type I are similar to those considered by Thoai [26] for $k = 2$. Problems of type II are similar to those considered by Kuno *et al.* [17]. Their parameters are defined as follows:

- $\alpha^0, \alpha^1, \ldots, \alpha^k$ are randomly generated vectors with all components belonging to $[0, 1]$.
- $A = (a_{ij}) \in \mathbb{R}^m \times \mathbb{R}^n$ is a randomly generated matrix with elements belonging to $[-1, 1]$.
- $b = (b_1, b_2, \ldots, b_m)$ is a randomly generated vector such that

$$b_i = \sum_{j=1}^{n} a_{ij} + 2b_0$$

  with b_0 being a randomly generated real in $[0, 1]$ for $i = 1, 2, \ldots, m$.
- $D^i \in \mathbb{R}^n \times \mathbb{R}^n$ are diagonal matrices with diagonal elements d^i_j randomly generated in [0,1].

The two programs have been implemented in C and run on a SUN-SPARC10/51 station (135.5 Mips, 27.3 Mflops, 64 Mram). We use the packages of CPLEX [3] for solving the linear programs and MINOS [20] (coded in FORTRAN) for the convex ones. Most of the time, the optimal solution of the previously solved (linear or convex) subproblem was used as starting point for the current subproblem (only one over 1000 subproblems was solved from the beginning).

The precision ε was set to 10^{-6}. However, as suggested in Section 4.2, if f_0 is the null function (i.e., for problems of types I and II), the precision is evaluated with respect to the reduced form φ' defined in (39).

Also, if $k = 2$, we subdivided the cones K using the points $w(K)$ rather than use bisections of ratio $\frac{1}{2}$.

Table I. Test Problems

Type I: $\min \prod_{i=1}^{k}(\alpha^i x)$ subject to : $Ax \leq b$, $x \geq 0.$	Type I0: $\min \alpha^0 x + \prod_{i=1}^{k}(\alpha^i x)$ subject to : $Ax \leq b$, $x \geq 0.$
Type II: $\min(\alpha^1 x) \prod_{i=2}^{k}(\alpha^i x + x^t D^i x)$ subject to : $Ax \leq b$, $x \geq 0.$	Type II0: $\min \alpha^0 x + (\alpha^1 x) \prod_{i=2}^{k}(\alpha^i x + x^t D^i x)$ subject to : $Ax \leq b$, $x \geq 0.$
	Type III0: $\min(\alpha^0 x + x^t D^0 x) + \prod_{i=1}^{k}(\alpha^i x + x^t D^i x)$ subject to : $\left\lvert x_1 + \sum_{j=2}^{k} \frac{j-1}{j} x_j \right\rvert^{1.5} \leq 1000$, $\left(-2 + \sum_{j=1}^{k} \frac{x_j}{j}\right)^2 \leq 100$, $Ax \leq b$, $x \geq 0.$

Table II. Number of iterations (*nb_iter*) for the cutting plane lower bounds

series	2,120,120	3,120,120	4,120,120
SOLQCM1a	23.2	270.8	8066.0
SOLQCM1b	23.2	250.9	4001.7
SOLQCM1c	22.9	233.4	2726.2

Finally, to easily access both the cone of smallest lower bound (Step 1 of the algorithm) and of greatest lower bound (Step 3), we used a min-max heap (see Atkinson *et al*. [1]) to store the cones.

For each set (= series) of parameters k, m, n, we solved 10 problems.

Table II compares the number of iterations in algorithm SOLQCM for the three variants of the first lower bound (see Section 3.3). The 10 problems considered are of type I. We observe that the best performances are obtained for SOLQCM1c. Other experiments have shown that the results are not better if the hyperplane is

Table III. Problems of type I.

	series	nb_iter		max_c		nb_conv		cpu_tot	
	(k, m, n)	μ	σ	μ	σ	μ	σ	μ	σ
	2,80,100	23.0	1.56	3.3	0.48	76.0	4.69	5.20	0.21
	2,100,100	22.6	1.07	3.4	0.52	74.8	3.22	7.32	0.63
S	2,100,120	22.4	2.12	3.1	0.32	74.2	6.36	9.32	0.56
O	2,120,120	22.9	2.13	3.4	0.84	75.7	6.39	13.43	1.53
L									
Q	3,80,100	249.4	19.96	22.0	2.58	758.2	59.87	25.74	4.78
C	3,100,100	247.0	21.56	21.0	2.91	751.0	64.68	34.72	3.76
M	3,100,120	258.6	26.95	21.9	2.85	785.8	80.85	44.39	8.59
1	3,120,120	233.4	16.79	19.6	2.37	710.2	50.38	56.33	10.51
c	3,120,140	256.6	27.60	22.0	2.67	779.8	82.81	74.39	13.90
	3,150,140	225.6	11.35	18.3	2.06	686.8	34.06	98.19	13.67
	3,150,160	237.3	18.28	20.1	3.03	721.9	54.83	125.07	20.91
	3,200,180	230.7	13.27	18.0	1.56	702.1	39.80	241.94	19.23
	4,80,100	3046.4	364.64	218.6	42.54	9152.2	1093.93	250.90	37.56
	4,100,100	2780.7	252.34	194.7	18.11	8355.1	757.01	338.29	62.56
	4,100,120	2953.1	373.04	216.1	19.99	8872.3	1119.12	456.20	59.17
	4,120,120	2726.2	307.25	180.1	19.01	8191.6	921.77	516.68	67.53
	5,80,100	124633.6	15190.68	9352.2	1369.42	373916.8	45572.03	11628.49	3015.24
SOL	2,80,100	13.8	1.62	1.7	0.48	32.6	3.24	3.91	0.32
QCM	2,100,100	13.0	1.25	1.6	0.52	31.0	2.49	5.96	1.57
2	2,100,120	13.3	1.34	1.8	0.42	31.6	2.67	7.52	0.83
	2,120,120	13.4	2.07	1.7	0.67	31.8	4.13	10.10	0.81

chosen to minimize the volume of the set $\mathcal{S}$ (see [6]). In the sequel, we consider only this variant c) of the first lower bound method.

Tables III and VII shows the results for problems of types I, II , I0, II0 and III0 respectively.

We observe that for $k = 2$, SOLQCM2 outperforms SOLQCM1c with respect to all indicators and for each size and type of problems. In particular, the number of convex subproblems to be solved (*nb_conv*) is everywhere more than half less for SOLQCM2 than for SOLQCM1c. Also for the problems of types I0, II0 and III0 (Tables V, VI and VII), the computing time *cpu_tot* is divided by a factor 2.

As noted by other authors, the results are very sensitive to the size k of the reduced (in this paper) quasiconcave optimization problem. For fixed k, the number of iterations *nb_iter*, the maximum number of cones in the min-max heap *max_c* (i.e., the maximum number of subproblems stored at any iteration) and the number of convex subproblems *nb_conv* do not increase significantly with the size $m \times n$,

Table IV. Problems of type II.

	series	nb_iter		max_c		nb_conv		cpu_tot	
	(k, m, n)	μ	σ	μ	σ	μ	σ	μ	σ
	2,80,100	26.3	1.06	3.6	0.52	85.9	3.18	42.63	6.77
	2,100,100	26.0	1.49	3.8	0.42	85.0	4.47	54.65	8.70
S	2,100,120	26.8	1.32	3.9	0.57	87.4	3.95	79.80	6.83
O	2,120,120	25.7	1.77	3.2	0.42	84.1	5.30	101.17	11.51
L									
Q	3,80,100	294.4	23.05	22.4	1.65	892.2	69.14	348.03	80.76
C	3,100,100	292.3	18.90	21.6	1.43	885.9	56.69	496.15	96.56
M	3,100,120	307.9	26.71	22.6	2.63	932.7	80.13	732.03	151.27
1	3,120,120	302.8	26.08	22.3	1.83	917.4	78.25	775.07	76.97
c									
	4,30,20	3304.7	457.81	247.8	50.25	9925.1	1373.43	227.78	46.94
	4,50,40	3342.7	399.81	230.6	47.73	10039.1	1199.44	854.99	195.13
	4,50,60	3574.9	346.85	255.0	46.94	10735.7	1040.55	1430.05	267.59
	4,60,80	3975.8	616.42	291.8	66.51	11938.4	1849.27	3200.88	1011.47
SOL	2,80,100	16.4	1.78	2.1	0.32	37.8	3.55	23.94	2.60
QCM	2,100,100	15.2	0.92	2.0	0.00	35.4	1.84	32.28	4.50
2	2,100,120	16.8	1.55	2.1	0.32	38.6	3.10	47.96	4.26
	2,120,120	15.3	0.82	2.0	0.00	35.6	1.65	61.36	6.60

Table V. Problems of type I0.

	series	nb_iter		max_c		nb_conv		cpu_tot	
	(k, m, n)	μ	σ	μ	σ	μ	σ	μ	σ
S	2,80,100	26.6	1.65	3.2	0.42	166.6	9.88	38.32	7.82
O	2,100,100	26.8	1.32	3.3	0.48	167.8	7.90	55.82	6.18
L	2,100,120	27.3	1.70	3.6	0.70	170.8	10.22	78.81	12.67
Q	2,120,120	27.8	2.15	3.7	0.82	173.8	12.90	107.13	19.06
C									
M	3,80,100	331.9	30.67	23.2	3.36	2665.2	245.35	480.26	52.49
1	3,100,100	311.8	23.65	20.0	1.94	2504.4	189.23	628.58	72.17
c	3,100,120	346.1	46.69	23.2	4.16	2778.8	373.53	887.67	184.92
	3,120,120	318.1	24.01	20.2	1.75	2554.8	192.05	1124.14	143.25
SOL	2,80,100	15.1	1.97	1.5	0.53	66.4	7.88	17.87	4.60
QCM	2,100,100	14.7	1.42	1.4	0.52	64.8	5.67	24.58	3.51
2	2,100,120	16.1	2.13	2.1	0.32	70.4	8.53	39.36	6.29
	2,120,120	16.7	2.00	2.0	0.67	72.8	8.01	49.00	7.53

Table VI. Problems of type II0.

	series	nb_iter		max_c		nb_conv		cpu_tot	
	(k, m, n)	μ	σ	μ	σ	μ	σ	μ	σ
S	2,80,100	31.4	1.65	4.2	0.63	195.4	9.88	81.12	13.06
O	2,100,100	29.9	1.10	3.6	0.52	186.4	6.60	104.75	15.47
L	2,100,120	31.4	1.43	3.6	0.52	195.4	8.58	145.92	26.19
Q	2,120,120	31.3	1.70	3.7	0.48	194.8	10.22	189.97	29.36
C									
M	3,80,100	427.8	33.92	23.0	1.89	3431.4	271.39	1378.39	304.81
1	3,100,100	414.6	19.68	22.8	1.32	3325.8	157.41	1675.58	233.25
c	3,100,120	438.1	35.10	22.9	1.10	3513.8	280.78	2359.82	420.98
	3,120,120	407.0	19.17	22.0	1.56	3265.0	153.33	2762.34	341.22
SOL	2,80,100	19.4	2.50	2.0	0.00	83.6	10.01	33.84	5.05
QCM	2,100,100	17.5	0.71	2.0	0.00	76.0	2.83	43.83	5.15
2	2,100,120	18.8	1.87	2.0	0.00	81.2	7.49	61.62	10.72
	2,120,120	18.4	1.58	2.0	0.00	79.6	6.31	79.18	7.28

Table VII. Problems of type III0.

	series	nb_iter		max_c		nb_conv		cpu_tot	
	(k, m, n)	μ	σ	μ	σ	μ	σ	μ	σ
	2,80,100	29.9	1.66	3.4	0.52	185.4	9.98	533.75	66.11
S	2,100,100	30.3	1.16	3.3	0.48	187.8	6.96	646.79	67.89
O	2,100,120	31.2	2.04	3.7	0.48	193.2	12.26	1016.63	100.68
L	2,120,120	31.3	1.06	3.4	0.70	193.8	6.36	1122.39	132.61
Q									
C	3,80,100	351.2	13.70	19.6	2.22	2817.6	109.61	7013.92	894.30
M	3,100,100	349.9	12.79	20.6	2.72	2807.2	102.34	7856.17	990.56
1	3,100,120	358.4	16.41	20.2	1.93	2875.2	131.25	12704.01	1366.17
c	3,120,120	371.1	18.44	20.7	1.16	2976.8	147.49	13816.79	1532.32
	4,30,20	3505.3	217.33	182.4	18.81	35063.0	2173.35	3589.84	714.44
	4,50,40	3651.2	202.22	185.1	6.56	36522.0	2022.17	14382.77	2059.55
SOL	2,80,100	19.3	1.64	1.8	0.42	82.2	6.55	258.60	35.18
QCM	2,100,100	18.7	1.83	1.8	0.42	79.8	7.32	312.41	28.25
2	2,100,120	19.0	1.70	1.9	0.32	81.0	6.80	449.98	50.03
	2,120,120	19.6	2.17	1.9	0.32	83.4	8.68	541.06	49.03

thus the computing time *cpu_tot* is essentially proportional to the time needed to solve a convex problem of the same size. We use μ to denote the average values, and σ to denote the standard deviations.

When comparing for example Tables III and V, we observe that the number of iterations *nb_iter* remains about the same but that the number of convex subproblems *nb_conv* and the computing time *cpu_tot* increase more dramatically. The increase of the number of convex subproblems can be mainly explained by the fact that the homogeneity property cannot be used anymore for problems of type I0. This implies in turn an increase of the computing time, which can also be explained by the fact that for problems of type I the evaluation of φ involves a linear program solved by CPLEX while for problems of type I0 it involves a convex nonlinear program solved by MINOS.

The results obtained in Table VII show that the addition of two convex constraints to a set of linear constraints together with the introduction of a convex function f_0 already increase the computing times by a factor of about 6 for algorithm SOLQCM1c and about 7 for algorithm SOLQCM2.

It is not easy to compare these results with those of the literature since the experiments are made on different problems and on different machines, e.g., Thoai [26] solved particular instances of problems similar to those of type I with $(k, m, n) = (2, 70, 120)$ within about 1950 seconds on a IBM-PS2 (Modell 88, with programs written in FORTRAN; Kuno *et al.* [17] solved problems similar to those of type II with $(k, m, n) = (3, 200, 180)$ within 914 seconds on average on a SUN4/75 (with programs written in C).

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